THE MINIMUM DISTANCE OF SETS OF POINTS AND THE MINIMUM SOCLE DEGREE

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ABSTRACT. Let $\mathbb{K}$ be a field of characteristic 0. Let $\Gamma \subset \mathbb{P}^n_{\mathbb{K}}$ be a reduced finite set of points, not all contained in a hyperplane. Let $hyp(\Gamma)$ be the maximum number of points of Γ contained in any hyperplane, and let $d(\Gamma) = |\Gamma| - hyp(\Gamma)$. If $I \subset R = \mathbb{K}[x_0, \ldots, x_n]$ is the ideal of Γ, then in [12] it is shown that for $n = 2, 3$, $d(\Gamma)$ has a lower bound expressed in terms of some shift in the graded minimal free resolution of R/I. In these notes we show that this behavior is true in general, for any $n \geq 2$: $d(\Gamma) \geq A_n$, where $A_n = \min\{a_i - n\}$ and $\oplus_i R(-a_i)$ is the last module in the graded minimal free resolution of R/I. In the end we also prove that this bound is sharp for a whole class of examples due to Juan Migliore ([10]).

1. INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic zero and let $\Gamma = \{P_1, \ldots, P_m\} \subset \mathbb{P}^n_{\mathbb{K}}$ be a reduced finite set of points, not all in a hyperplane (i.e., non-degenerate). Let $hyp(\Gamma)$ be the maximum number of points of Γ lying in any hyperplane. Define *the minimum distance of the set* Γ to be the number

$$d(\Gamma) = m - hyp(\Gamma).$$

The reason we borrowed this terminology from coding theory is that $d(\Gamma)$ is exactly the minimum distance of the (equivalence class of) linear codes with generating matrix having as columns the coordinates of the points of Γ (see [14] for more details).

Denote with $R = \mathbb{K}[x_0, \ldots, x_n]$ the (homogeneous) ring of polynomials with coefficients in $\mathbb{K}$. Let $I \subset R$ be the ideal of Γ. The goal of these notes is to study $d(\Gamma)$ using the graded minimal free resolution of R/I.

Some preliminary results were obtained in [6] when Γ is a complete intersection, and generalized in [12] when Γ is (arithmetically) Gorenstein. In both situations

$$d(\Gamma) \geq reg(R/I),$$

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the Castelnuovo-Mumford regularity. The question became if this lower bound is true for any reduced non-degenerate finite set of points ([13]). As we will see below (Example 2.5), the answer is negative, yet we will still be able to give a lower bound for $d(\Gamma)$ in this general setup, in terms of the shifts in the graded minimal free resolution of R/I.

If $A = \oplus_{i=0} A_i$ is a graded Artinian $\mathbb{K}-$algebra with maximal ideal $\underline{m} = \oplus_{i>0} A_i$, then $soc(A) = 0 : \underline{m}$ is a finite dimensional graded $\mathbb{K}-$vector space, called *the socle of* A. So

$$soc(A) = \oplus \mathbb{K}(-b_i),$$

and the positive integers b_i are called *the socle degrees of* A.

In our case, if $\bar{I}$ is the Artinian reduction of I, the ideal of Γ, and if

$$0 \to F_n = \oplus R(-a_i) \to \cdots \to F_1 \to R \to R/I \to 0$$

is the graded minimal free resolution of R/I, then the last module in the free resolution of $A = R/\bar{I}$ is $F_n(-1) = \oplus R(-(a_i+1))$ and sits in position $n+1$. So, by [9], Lemma 1.3, the socle degrees of A are exactly

$$b_i = (a_i + 1) - (n+1) = a_i - n.$$

We'll abuse the terminology by saying that the socle degrees of $A = R/\bar{I}$ are the socle degrees of R/I.

Denote

$$A_n = \min\{a_i - n\}$$

to be the minimum value of the socle degrees.

In [12], Theorem 4.1, we showed that if Γ is any reduced non-degenerate finite set of points in $\mathbb{P}^k, k = 2, 3$, then $d(\Gamma) \geq A_k$. In the first part of these notes we generalize this result (Theorem 2.4) showing that if Γ is any reduced non-degenerate finite set of points in $\mathbb{P}^n, n \geq 2$, then

$$d(\Gamma) \geq A_n,$$

and in the second part we investigate if this bound is sharp.

2. A LOWER BOUND ON THE MINIMUM DISTANCE OF SETS OF POINTS

Let $\Gamma = \{P_1, \ldots, P_m\} \subset \mathbb{P}^n$ be a reduced non-degenerate finite set of points. We denoted with $hyp(\Gamma)$ the maximum number of points of Γ contained in any hyperplane. To obtain the maximum number of points of Γ contained in any hypersurface of degree a, by [11], one should compute $hyp(v_a(\Gamma))$, where v_a is the Veronese embedding of degree a of $\mathbb{P}^n$ into $\mathbb{P}^{N_a}$, where $N_a = \binom{n+a}{a} - 1$. Let us denote

$$d(\Gamma)_a = |\Gamma| - hyp(v_a(\Gamma)).$$

Observe that $d(\Gamma)_1 = d(\Gamma)$.

From [13] (using [7]), $d(\Gamma)_a$ is the minimum distance of the evaluation code of order a associated to Γ. With this fact in mind, [12], Proposition 2.1, will constitute the key tool to prove our main result:

Lemma 2.1. *(*[12]*) If $d(\Gamma)_b \geq 2$ for some $b \geq 2$, then for all $1 \leq a \leq b-1$, we have $d(\Gamma)_a \geq d(\Gamma)_{a+1} + 1$. Therefore, if $d(\Gamma)_b \geq 2$ for some $b \geq 1$, we have $d(\Gamma)_a \geq b - a + 2$ for all $1 \leq a \leq b$.*

In general, if $a \leq b$ then $d(\Gamma)_a \geq d(\Gamma)_b$.

Let $\Gamma' = \Gamma \setminus \{P_m\}$. Let $I = I(\Gamma)$ and $I' = I(\Gamma')$ be the homogeneous ideals in $R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ of the sets Γ and Γ'.

Since $\Gamma' \subsetneq \Gamma$, then $I \subsetneq I'$, and consider

$$\delta(P_m) = \min\{d | \dim(I'_d) > \dim(I_d)\} \geq 1.$$

An element in $I' \setminus I$ is called a *separator* of P_m, and $\delta(P_m)$ is called the *degree of the point P_m in* Γ. By [4], the Hilbert function of the R/I and the degree of a point in Γ are related by the following formula:

Lemma 2.2. *(*[4]*)*

$$HF(R/I, i) = \begin{cases} HF(R/I', i), & \textit{if } 0 \leq i \leq \delta(P_m) - 1; \\ HF(R/I', i) + 1, & \textit{if } i \geq \delta(P_m). \end{cases}$$

Suppose the graded minimal free resolution of the $R-$ module R/I is

$$0 \to F_n = \oplus R(-a_i) \to \cdots \to F_1 \to R \to R/I \to 0,$$

and let $A_n = \min\{a_i - n\}$ be the minimum socle degree of R/I.

It was shown in [1], for the case of points in $\mathbb{P}^2$, and, in general, in [2] (using [8]), for the case of points in $\mathbb{P}^n, n \geq 2$, that the degree of a point in Γ is among the socle degrees of R/I.

Lemma 2.3. *(*[2]*) If P is any point in Γ and $\delta(P)$ is as above, then*

$$\delta(P) \geq A_n.$$

Once we have this, we can prove the main result.

Theorem 2.4. *In the above notations,*

$$d(\Gamma) \geq A_n.$$

Proof. The set Γ is non-degenerate, so $A_n \geq 1$. If $A_n = 1$, then the result is immediate since $d(\Gamma) \geq 1$ all the time. Assume that $A_n \geq 2$.

Let

$$\delta = \delta(\Gamma) = \min\{\delta(P_i) | i = 1, \ldots, m\}.$$

If $\delta = 1$, then from Lemma 2.3 $A_n = 1$. So let us assume that $\delta \geq 2$ and consider $d(\Gamma)_{\delta-1}$.

By [7], for any $a \geq 1$, we have that

$$d(\Gamma)_a = |\Gamma| - \max_{\Gamma' \subset \Gamma}\{|\Gamma'| : \dim(I(\Gamma')_a) > \dim(I(\Gamma)_a)\}.$$

So, if $d(\Gamma)_{\delta-1} = 1$, then there exists $Q \in \Gamma$ such that $\dim(J_{\delta-1}) > \dim(I_{\delta-1})$, where J is the ideal of $\Gamma \setminus \{Q\}$. From Lemma 2.2,

$$\delta - 1 \geq \delta(Q).$$

But this contradicts the minimality of δ. Therefore,

$$d(\Gamma)_{\delta-1} \geq 2.$$

From Lemma 2.3 we have that $\delta - 1 \geq A_n - 1$ and therefore,

$$d(\Gamma)_{A_n-1} \geq d(\Gamma)_{\delta-1} \geq 2.$$

By using Lemma 2.1 with $b = A_n - 1$ and $a = 1$, we obtain

$$d(\Gamma) = d(\Gamma)_1 \geq (A_n - 1) - 1 + 2 = A_n.$$

□

Example 2.5. Consider $\Gamma = \{[0,0,1],[0,1,0],[0,2,1],[0,3,1],[1,0,0]\} \subset \mathbb{P}^2$. The first four points lie on the line of equation $x = 0$, and the fifth does not. Therefore $hyp(\Gamma) = 4$ and $d(\Gamma) = 5 - 4 = 1$. The ideal of Γ in $R = \mathbb{K}[x,y,z]$ is

$$I = \langle x,y \rangle \cap \langle x,z \rangle \cap \langle x, 2z-y \rangle \cap \langle x, 3z-y \rangle \cap \langle y,z \rangle.$$

With the help of Macaulay 2 by Grayson and Stillman, the minimal graded free resolution of R/I is:

$$0 \to R(-5) \oplus R(-3) \to R(-4) \oplus R^2(-2) \to R \to R/I \to 0.$$

So $reg(R/I) = 5 - 2 = 3$ and $A_2 = 3 - 2 = 1$.

3. Sets of points with minimum distance equal to A_n

Example 2.5 belongs to the class of examples for which $d(\Gamma) = A_n$. In this section we are going to investigate the following question: for given n and m, under what conditions we can find, if it exists, a non-degenerate reduced finite set of m points $\Gamma \subset \mathbb{P}^n$ with $d(\Gamma) = A_n$? Also we can ask a bit more: for given n, m and $d(\Gamma)$, can we construct a non-degenerate reduced finite set of m points $\Gamma \subset \mathbb{P}^n$ with $d(\Gamma) = A_n$?

Denote with $a(\Gamma) = \min\{a_i\}$ (we keep the same notations as before: $F_n = \oplus R(-a_i)$ is the last module in the graded minimal free resolution of R/I). Therefore, $A_n = a(\Gamma) - n$.

First of all, since $R(-a(\Gamma))$ is a direct summand in F_n, then $a(\Gamma) \geq n$. If $a(\Gamma) = n$, then one will have $R(-1)$ as a direct summand in F_1, which

means that I has a minimal generator of degree 1. This means that Γ lies in a hyperplane and, therefore, Γ is degenerate. So we must have that

$$a(\Gamma) \geq n+1.$$

Let's see some simple cases:

Example 3.1. The case: $a(\Gamma) = n+1$. This is the case of Example 2.5. Construct Γ as $m-1$ points lying in a hyperplane and one point outside this hyperplane. From Theorem 2.4, since $d(\Gamma) = m-(m-1) = 1$, we have $a(\Gamma) - n \leq 1$ and from the restriction above, we have $a(\Gamma) = n+1$. So this set satisfies the requirement $d(\Gamma) = A_n$.

Example 3.2. The case $a(\Gamma) = n+2$. Since any n points in $\mathbb{P}^n$ lie in a hyperplane, then $m \geq n+2$ (if $m = n+1$ we'd be in the case above). If $m = n+2$, let's pick Γ to be a generic set of $n+2$ points in $\mathbb{P}^n$. By [5], R/I is Gorenstein of regularity $r = 2$. So $A_n = r = 2$. Since $hyp(\Gamma) = n$, we have that $d(\Gamma) = (n+2) - n = 2 = A_n$.

In general, let us consider the following set of points Γ in $\mathbb{P}^n$, suggested by Juan Migliore ([10]).

Let $\Gamma_1 \subset \mathbb{P}^n$ be a generic set of α points in $\mathbb{P}^{n-1}$ embedded in $\mathbb{P}^n$ (assume the hyperplane where they lie has equation $x_0 = 0$).

Let $\Gamma_2 \subset \mathbb{P}^n$ be a set of β distinct points on a line in $\mathbb{P}^n$ not contained in the above hyperplane. Assume that the coordinates of these points are $[1, u_i, 1 \ldots, 1], 1 \leq i \leq \beta, u_i \neq u_j$.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and we would like to have that $hyp(\Gamma) = \alpha$ (so one immediate restriction is that $\alpha \geq \beta + n - 2$).

The goal is to see under what conditions

$$d(\Gamma) = (\alpha+\beta) - \alpha = \beta = A_n.$$

Let $I, I_1, I_2 \subset R = \mathbb{K}[x_0, \ldots, x_n]$ be the ideals of the sets Γ, Γ_1 and, respectively, Γ_2.

We have that

$$I_2 = \langle \prod_{i=1}^{\beta} (u_i x_0 - x_1), x_2 - x_0, \ldots, x_n - x_0 \rangle$$

and

$$I_1 = \langle x_0, J \rangle,$$

where $J \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is the ideal of the generic set of α points in $\mathbb{P}^{n-1}$.

First, let s be the smallest integer such that $\alpha < \binom{s+n-1}{n-1}$. Since J is the ideal of a generic set of α points in $\mathbb{P}^{n-1}$, then the Hilbert function is as nice as possible (in fact this is the definition of a generic set of points):

$$HF(S/J,i) = \begin{cases} \binom{i+n-1}{n-1}, & \text{if } i \leq s-1; \\ \alpha, & \text{if } i \geq s. \end{cases}$$

Suppose the minimal free resolution of S/J is

$$0 \to C_{n-1} \to \cdots \to C_1 \to S \to S/J \to 0.$$

Suppose that u is the minimum shift in C_{n-1}. Then $u-(n-1) \geq s$; otherwise, moving down on the resolution to C_1 we'd have an element of degree $< s$ and this contradicts the Hilbert function. Also the Hilbert function tells us that the regularity of S/J is s. So S/J is *level*:

$$C_{n-1} = S^k(-(s+n-1)).$$

J is minimally generated in degree $\geq s$ and the regularity of S/J is s, therefore

$$C_1 = S^{p_1}(-s) \oplus S^{p_2}(-(s+1)).$$

Since $I_1 = \langle x_0, J\rangle$, then the minimal free resolution of R/I_1 is:

$$\mathbb{G}_* : 0 \to G_n = C_{n-1}[x_0](-1) \to G_{n-1} = C_{n-2}[x_0](-1) \oplus C_{n-1}[x_0] \to \cdots$$
$$\to G_1 = R(-1) \oplus C_1[x_0] \to R \to R/I_1 \to 0,$$

where if $C_i = \oplus S(-c_{ij})$, we denoted $C_i[x_0] = \oplus R(-c_{ij})$.

Also, since J is the ideal of points not all lying in a hyperplane, then $J \nsubseteq \langle x_2, \ldots, x_n\rangle$, and therefore one can assume that

$$\langle J, x_2, \ldots, x_n\rangle = \langle x_1^v, x_2, \ldots, x_n\rangle,$$

for $v = s$ or $v = s+1$.

We have that $I = I_1 \cap I_2$ which leads to the following exact sequence of $R-$modules:

$$(*)\ 0 \to R/I \to R/I_1 \oplus R/I_2 \to R/(I_1+I_2) \to 0.$$

We have that

$$I_1 + I_2 = \langle x_0, J, \prod_{i=1}^{\beta}(u_i x_0 - x_1), x_2 - x_0, \ldots, x_n - x_0\rangle$$
$$= \langle x_0, x_1^t, x_2, \ldots, x_n\rangle,$$

where $t = \min\{v, \beta\}$.

With this, $I_1 + I_2$ is a complete intersection of codimension $n+1$ and $R/(I_1+I_2)$ has minimal free resolution

$$\mathbb{E}_* : 0 \to E_{n+1} = R(-(t+n)) \to \cdots \to E_1 = R(-t) \oplus R(-1)^n \to R.$$

Also I_2 is a complete intersection of codimension n and R/I_2 has minimal free resolution

$$\mathbb{H}_* : 0 \to H_n = R(-(\beta+n-1)) \to \cdots \to H_1 = R(-\beta) \oplus R(-1)^{n-1} \to R.$$

Suppose the minimal free resolution of R/I is

$$\mathbb{F}_* : 0 \to F_n \to \cdots \to F_1 \to R \to R/I \to 0.$$

The mapping cone construction (see [3] for background on resolutions) applied to the exact sequence $(*)$ above gives the following free resolution (not necessarily minimal) for $R/(I_1 + I_2)$:

$$\mathbb{W}_* : 0 \to W_{n+1} = F_n \to W_n = F_{n-1} \oplus (G_n \oplus H_n) \to \cdots$$
$$\to W_1 = R \oplus G_1 \oplus H_1 \to R^2 \to R/(I_1 + I_2) \to 0.$$

Comparing this with the minimal free resolution we obtained before we get that $E_{n+1} = R(-(t + n))$ is a direct summand of $W_{n+1} = F_n$. So $t + n \geq a(\Gamma)$ and hence,

$$t \geq A_n.$$

This leads to the following restriction:

Lemma 3.3. *If $s \leq \beta - 2$, then*

$$A_n < \beta.$$

Proof. If $s \leq \beta - 2$, then $t = \min\{v, \beta\} < \beta$. □

$\mathbb{W}_*$ is a free resolution of $R/(I_1 + I_2)$ and $\mathbb{E}_*$ is a minimal free resolution of the same $R-$module $R/(I_1 + I_2)$. From the definition of minimality, one can obtain $\mathbb{E}_*$ from $\mathbb{W}_*$ by removing the redundancies in $\mathbb{W}_*$; that is, some differential maps in $\mathbb{W}_*$ have pieces of degree 0 that can be erased. This process of removing the redundancies will be called a *cancellation*. For example, in the differential

$$W_1 = R \oplus G_1 \oplus H_1 \to R^2,$$

we have the redundancy $R \to R$ that can be removed to obtain

$$G_1 \oplus H_1 \to R.$$

Lemma 3.4. *If $s \geq \beta$, then*

$$A_n = \beta \text{ or } A_n = \beta - 1.$$

Proof. If $s \geq \beta$, then since $v = s$ or $s + 1$ we have that $t = \min\{v, \beta\} = \beta$. We saw right before Lemma 3.3 that

$$A_n \leq t = \beta$$

and

$$W_{n+1} = F_n = R(-(\beta+n)) \oplus K.$$

The only way one has a cancellation in W_{n+1} to obtain $E_{n+1} = R(-(\beta+n))$ is only if K is a direct summand in

$$W_n = F_{n-1} \oplus (G_n \oplus H_n).$$

But K is a direct summand in F_n and $0 \to F_n \to F_{n-1}$ is a part of a minimal free resolution, so there are no cancellations possible here. Therefore, K is a direct summand in

$$G_n \oplus H_n = R^k(-(s+n)) \oplus R(-(\beta+n-1)).$$

If $A_n \neq \beta$ then $A_n < \beta$ and so $a(\Gamma) = A_n + n < \beta + n$. So $R(-a(\Gamma))$, which is a direct summand in F_n, should occur as a direct summand in K. So $a(\Gamma) = s+n$ or $a(\Gamma) = n+\beta-1$. Since $s \geq \beta$ we have $a(\Gamma) < \beta+n \leq s+n$ and we are left with

$$A_n = \beta - 1.$$

$\square$

Lemma 3.5. *If $s \geq \beta+2$ then*

$$A_n = \beta.$$

Proof. We have $s \geq \beta+2$. Again $t = \beta$ and let's assume that $A_n = \beta - 1$. From the proof of Lemma 3.4, since $A_n = \beta - 1$ and therefore $a(\Gamma) = \beta + n - 1$, we have that

$$K = R^p(-(s+n)) \oplus R(-(\beta+n-1)),$$

for some $p \leq k$. So we have

$$F_n = R^p(-(s+n)) \oplus R(-(\beta+n)) \oplus R(-(\beta+n-1)).$$

We must mention that we used the one copy of $R(-(\beta+n-1))$ to obtain the corresponding cancellation in W_{n+1} that gave us $E_{n+1} = R(-(\beta+n))$.

To obtain $E_n = R^n(-(\beta+n-1)) \oplus R(-n)$ from $W_n = F_{n-1} \oplus R^k(-(s+n)) \oplus R(-(\beta+n-1))$ through a cancellation, since we already used $R(-(\beta+n-1))$ and since $s \geq \beta+2$, then the whole block $R^n(-(\beta+n-1)) \oplus R(-n)$ should be a direct summand inside F_{n-1}.

We have that

$$\mathcal{A} = \{x_0(x_2-x_0), \ldots, x_0(x_n-x_0), x_0\prod_{i=1}^{\beta}(u_ix_0-x_1)\}$$

is a subset of the minimal generators of I. In fact

$$F_1 = R^{n-1}(-2) \oplus R(-(\beta+1)) \oplus \bigoplus R(-a_{1j}).$$

Claim: $\min\{a_{1j}\} \geq s$.

Proof of Claim: Let $f \in I = I_1 \cap I_2$, with $\deg(f) = b < s$. Since $f \in I_1 = \langle x_0, J\rangle$, then we can assume that $f = x_0 g + h, g \in R$ and $h \in J \cap \mathbb{K}[x_1, \ldots, x_n]$ with $\deg(h) = b$. Since J is minimally generated in degree $\geq s$, then $h = 0$ and we get that $f \in \langle x_0\rangle$. So $f \in \langle x_0\rangle \cap I_2$ and therefore, after the change of variables $x'_0 = x_0, x'_1 = x_1, x'_2 = x_2 - x_0, \ldots, x'_n = x_n - x_0$, we have that

$$f = x'_0 f_0 = x'_2 f_2 + \cdots + x'_n f_n + (\prod_{i=1}^{\beta}(u_i x'_0 - x'_1)) f_1,$$

where $f_i \in \mathbb{K}[x'_0, \ldots, x'_n]$.

We have that

$$ht(\langle x'_0, x'_2, \ldots, x'_n, \prod_{i=1}^{\beta}(u_i x'_0 - x'_1)\rangle) = ht(\langle x'_0, x'_2, \ldots, x'_n, (x'_1)^{\beta}\rangle) = n+1,$$

so $\{x'_0, x'_2, \ldots, x'_n, \prod_{i=1}^{\beta}(u_i x'_0 - x'_1)\}$ forms a regular sequence and so $f_0 \in \langle x'_2, \ldots, x'_n, \prod_{i=1}^{\beta}(u_i x'_0 - x'_1)\rangle$. This implies that

$$f = x'_0 f_0 \in \langle x'_0 x'_2, \ldots, x'_0 x'_n, x'_0 \prod_{i=1}^{\beta}(u_i x'_0 - x'_1)\rangle.$$

We just proved that if $f \in I$ of degree $\deg(f) < s$, then $f \in \langle \mathcal{A}\rangle$. So the Claim is shown.

	0	1	$\cdots$	$n-1$	n
total:	1	b_1	$\cdots$	b_{n-1}	b_n
0:	1	-	$\cdots$	-	-
1:	-	$n-1$	$\cdots$	1	-
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
β:	-	1	$\cdots$	$n-1$	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$s-1$:	-	c_1	$\cdots$	c_{n-1}	c_n
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$

The table above describes how the betti diagram of R/I should look like. It is important to mention that since $s \geq \beta + 2$, then all the syzygies of any order involving at least one minimal generator of I of degree $\geq s$ should occur in the row labeled $s-1$ or below. With this in mind, $R^n(-(\beta + n -$

$1)) \oplus R(-n)$ inside F_{n-1} can be obtained only from the (Koszul) syzygies on the set $\mathcal{A}$. But the $(n-1)-$syzygy module of $\mathcal{A}$ is

$$R^{n-1}(-(\beta+n-1)) \oplus R(-n).$$

So if $A_n = \beta - 1$, we get an extra $R(-(\beta+n-1))$ in F_{n-1}. Contradiction. Consequently, we must have $A_n = \beta$. $\square$

If we put everything together we have:

Theorem 3.6. *Let $\Gamma_1 \subset \mathbb{P}^n$ be a generic set of α points in a hyperplane in $\mathbb{P}^n$ and let $\Gamma_2 \subset \mathbb{P}^n$ be a set of β distinct points on a line in $\mathbb{P}^n$ not contained in this hyperplane. Suppose that $\alpha \geq \beta + n - 2$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$. Then:*

(1) *If $\alpha < \binom{\beta+n-3}{n-1}$, then $d(\Gamma) > A_n$.*
(2) *If $\alpha \geq \binom{\beta+n}{n-1}$, then $d(\Gamma) = A_n$.*

Proof. Since s is the smallest integer such that $\alpha < \binom{s+n-1}{n-1}$, then $\alpha < \binom{\beta+n-3}{n-1}$ will give us that $s \leq \beta - 2$. Similarly, $\alpha \geq \binom{\beta+n}{n-1}$ implies that $s > \beta + 1$. We obtain the theorem by using Lemma 3.3 and Lemma 3.5 above. $\square$

We end with some examples describing what can happen if s is in the range not covered by the theorem above: $s = \beta - 1, \beta, \beta + 1$. Keeping in mind that $d(\Gamma) = \beta$, we want to see if $d(\Gamma) = A_n$ or not.

Example 3.7. If $s = \beta$, then both situations in Lemma 3.4 can occur.

First, Example 3.2 belongs to this situation: $\alpha = n < \binom{2+n-1}{n-1}$ (so $s = 2$) and $\beta = 2$. For this example we have that $d(\Gamma) = A_n$.

Next, consider the following set of $\alpha = 6$ points contained in the hyperplane of $\mathbb{P}^3$ of equation $x_0 = 0$:

$$\Gamma_1 = \{[0,0,0,1], [0,1,0,1], [0,0,1,1], [0,1,1,1], [0,2,1,2], [0,-1,-2,1]\}.$$

Disregarding the first coordinate $x_0 = 0$, we have a set of $6 = \binom{2+2}{2}$ points in $\mathbb{P}^2$, and so $s = 3$. We have that the ideal $J \subset \mathbb{K}[x_1, x_2, x_3]$ of these points is minimally generated by four cubic generators. So these six points form a generic set of points in $\mathbb{P}^2$.

Consider the following set of $\beta = 3 = s$ points on a line in $\mathbb{P}^3$:

$$\Gamma_2 = \{[1,7,5,0], [1,3,4,0], [2,10,9,0]\}.$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and let $I \subset R = \mathbb{K}[x_0, x_1, x_2, x_3]$ be the ideal of Γ. With Macaulay 2 we can obtain the graded minimal free resolution of R/I:

$$0 \to R(-6) \oplus R(-5) \to R^6(-4) \oplus R(-3) \to R^4(-3) \oplus R^2(-2) \to R.$$

We have $A_3 = 5 - 3 = 2$, and therefore $d(\Gamma) = A_3 + 1$.

Example 3.8. In the previous example if we remove the last point from the set Γ_1, we are in the situation of a generic set of five points in the hyperplane $x_0 = 0$ in $\mathbb{P}^3$, with $s = 2$. Keeping the same Γ_2 as above (and so $s = \beta - 1$), we obtain that $d(\Gamma) = A_3 + 1$.

If in Example 3.7 we keep Γ_1 as is, and if we remove one point from Γ_2, we will be in the situation when $s = \beta + 1$. With Macaulay 2 we obtain that $d(\Gamma) = A_3$.

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